Lecture 6

Monday, October 14, 2019 6:27 AM

- Recall: Compactness, sequential compactness; Prove Lebesgue's covering lemma from Last Lecture 5 notes.

Thm 1. Let (X, d) be metric space. TFAE:

(a) X is compact.

(b) Every infinite set has a limit point.

(c) X is seq. compact

(d) X is complete and totally bounded – $\forall \varepsilon > 0\ \exists x_1, \dots, x_n$ s.t. $X \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon)$.

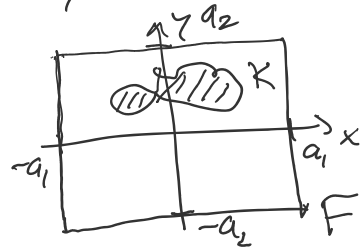
Clearly stronger than "bounded".

Pf. Suggested strategy $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$. Then $(c), (d) \Rightarrow (a)$.
Proof are all similar to pfs done earlier, w/ possible exception of $(c), (d) \Rightarrow (a)$, which uses Lebesgue's covering Lemma. We do this implication and refer to Conway for the rest.

$(c) \Rightarrow (a)$ (using $(c) \Leftrightarrow (d)$): Let $\{G_\alpha\}_{\alpha \in I}$ be open cover of X. By Lebesgue's Covering Lemma $\exists \varepsilon > 0$ s.t. $\forall x \in X$, $B(x, \varepsilon) \subseteq G_\alpha$, some α.
Since $(c) \Leftrightarrow (d)$, X is totally bounded $\exists x_1, \dots, x_n$ s.t. $X \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon)$.
But $B(x_k, \varepsilon) \subseteq G_{\alpha_k}$, so $X \subseteq \bigcup_{k=1}^{n} G_{\alpha_k}$, finite subcover $\Rightarrow$ X compact. ∎

Heine-Borel Thm. Let $K \subseteq \mathbb{R}^n$ (e.g. $\mathbb{C} \cong \mathbb{R}^2$). K closed + bounded $\Leftrightarrow$ K is compact.

Pf. $\Rightarrow$: K bounded $\Rightarrow \exists (a_1, \dots, a_n) \in \mathbb{R}^n$, $a_j > 0$, s.t.
$K \subseteq F := [-a_1, a_1] \times \dots \times [-a_n, a_n]$,



Now, the closed cube F is complete as space ($F \subseteq X$ closed, X complete) and totally bounded (not completely obvious, but HW #2 problem).
By Thm 1, F is compact. Since $K \subseteq F$ is closed, K is compact (Basic Props).

$\Leftarrow$: K compact $\Rightarrow$ K closed (Basic Props). Also, by Thm 1, compact (Basic Props).

$\Leftarrow$: K compact $\Rightarrow$ K closed (Basic Props). Also, by Thm 1, K totally bdd $\Rightarrow$ K bdd. $\boxtimes$

Continuity.

Def. ① Let (X,d), (Ω,ρ) be metric spaces. A function $f: X\to\Omega$ is continuous at $x_0\in X$ if $\forall\varepsilon>0\ \exists\delta>0$ s.t. $\rho(f(x),f(x_0))<\varepsilon$ when $d(x,x_0)<\delta$. Write $\lim_{x\to x_0} f(x)=f(x_0)$.

② $f: X\to\Omega$ is continuous (on X) if f is continuous at x_0 for every $x_0\in X$.

Prop 1. Let $f: X\to\Omega$ be given. TFAE:

(a) f is cont.

(b) $\forall\,\Delta\subseteq\Omega$ open $f^{-1}(\Delta)\subseteq X$ is open.

(c) $\forall\,\Gamma\subseteq\Omega$ closed $f^{-1}(\Gamma)\subseteq X$ is closed.

Pf. (a)$\Rightarrow$(b): Let $a\in f^{-1}(\Delta)$, $\omega=f(a)\in\Delta$. Pick $\varepsilon>0$. Since Δ open $\exists\,B(\omega,\varepsilon)\subseteq\Delta$. Also, by cont., $\exists\,\delta>0$ s.t. $\rho(f(x),\omega)<\varepsilon$ ($\Leftrightarrow f(x)\in B(\omega,\varepsilon)$) when $d(x,a)<\delta$ ($\Leftrightarrow x\in B(a,\delta)$). In other words, $B(a,\delta)\subseteq f^{-1}(\Delta)\Rightarrow f^{-1}(\Delta)$ open.

(b)$\Rightarrow$(c): $\Omega\setminus\Gamma$ is open $\Rightarrow f^{-1}(\Omega\setminus\Gamma)$ open. But $f^{-1}(\Omega\setminus\Gamma)=X\setminus f^{-1}(\Gamma)$, so $f^{-1}(\Gamma)$ is closed by def.

$f^{-1}(\Sigma \setminus F) = X \setminus f^{-1}(F)$, so $f^{-1}(F)$ ...
def.

(c) $\Rightarrow$ (b): Same argument as above, swapping open and closed.

(b) $\Rightarrow$ (a): Pick $a \in X$, $\varepsilon > 0$. Set $\omega = f(a)$. $B(\omega, \varepsilon)$ is open $\Rightarrow$ $f^{-1}(B(\omega,\varepsilon))$ is open; $a \in f^{-1}(B(\omega,\varepsilon)) \Rightarrow \exists\, \delta > 0$ s.t. $B(a,\delta) \subseteq f^{-1}(B(\omega,\varepsilon))$; i.e. $\rho(f(x),\omega) < \varepsilon$ when $d(x,a) < \delta$. $\Rightarrow$ f cont. at a. But $a \in X$ arbitrary $\Rightarrow$ f cont. on X. ▨

Basic Prop. ① $f, g : X \to \mathbb{C}$ cont. Then, $f+g$, fg cont., and $f/g : X \setminus \{x : g(x) = 0\} \to \mathbb{C}$ is cont.

② If $X \xrightarrow{g} \Sigma \xrightarrow{f} \Omega$ w/ f, g cont. $\Rightarrow$ $f \circ g : X \to \Omega$ is cont.

Pf. ① Ex. ② Use Prop 1 (b) charact. of cont; $(f \circ g)^{-1}(\Delta) = g^{-1}(f^{-1}(\Delta))$. ▨